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Bosonization and Vertex Algebras with Defects

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Abstract

The method of bosonization is extended to the case when a dissipationless point-like defect is present in space-time. Introducing the chiral components of a massless scalar field, interacting with the defect in two dimensions, we construct the associated vertex operators. The main features of the corresponding vertex algebra are established. As an application of this framework we solve the massless Thirring model with defect. We also construct the vertex representation of the $\widehat{sl}(2)$ Kac-Moody algebra, describing the complex interplay between the left and right sectors due to the interaction with the defect. The Sugawara form of the energy-momentum tensor is also explored.

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1 Introduction

Bosonization represents a powerful method for solving a variety of models [1]-[4] in two space-time dimensions with relevant applications in both condensed matter physics and string theory. On the mathematical side bosonization is among the fundamental tools for constructing vertex algebras [5] and in particular, vertex representations of affine Kac-Moody algebras [6]. The main goal of the present paper is to extend the framework of bosonization to the case when defects (impurities) are present in space. The subject of defects attracts recently much attention in different areas of quantum physics, including quantum mechanics [7]-[9], integrable systems [10]-[20], conformal and finite temperature quantum field theory [21]-[23] and string theory, where branes can be considered as purely reflecting defects. Some interesting studies [25, 26] of both reflecting and transmitting branes (permeable conformal walls) are also worth mentioning. The new configurations we introduce and develop here concern chiral fields and vertex operators with defects. The study of these structures is motivated by potential physical applications to condensed matter physics and string theory. On the mathematical side our work suggests some interesting generalizations in the context of vertex algebras.

We focus below on dissipationless point-like defects, showing that they preserve the basic ingredients of bosonization - quantum field locality and unitarity. In section 2 we summarize some results about the massless scalar field and its dual interacting with a generic defect of the above type. The relative vertex operators are constructed and investigated in section 3. We show that they generally obey anyon statistics, thus describing anyon fields in the presence of defects. Sections 4 and 5 are devoted to some applications. In section 4 we solve the massless Thirring model with a point-like defect and discuss the solution. In section 5 we study some aspects of non-abelian bosonization with impurities. We describe there the vertex operator construction of the $\widehat{sl}(2)$ Kac-Moody algebra, focusing on the new features stemming from the presence of a defect. Adopting the Sugawara representation, we investigate also the impact of the defect on the energy-momentum tensor. Finally, section 6 contains our conclusions and some indications for further developments.

2 General setting

Bosonization (see e. g. [27]) has a long history dating back [28] to the earliest years of quantum field theory. The main building blocks are the massless scalar field $\varphi(t,x)$ and its dual $\widetilde{\varphi}(t,x)$. Therefore, our first step will be to establish the basic properties of $\{\varphi,\, \widetilde{\varphi}\}$ when a point-like dissipationless defect is present in space. Without loss of generality one can localize the defect at $x = 0$ and consider thus the following

equation of motion

$$\left(\partial_t^2 - \partial_x^2\right) \varphi(t,x) = 0\, , \qquad x \neq 0\, , \tag{2.1}$$

with standard initial conditions fixed by the equal-time canonical commutation relations

$$[\varphi(0,x_1)\, ,\, \varphi(0,x_2)] = 0\, , \qquad [(\partial_t\varphi)(0,x_1)\, ,\, \varphi(0,x_2)] = -i\delta(x_1-x_2)\, . \tag{2.2}$$

The most general dissipationless interaction of $\varphi(t,x)$ with the defect at $x = 0$ is described [8] by the boundary condition

$$\left(\begin{array}{c} \varphi(t,+0) \\ \partial_x\varphi(t,+0) \end{array}\right) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{c} \varphi(t,-0) \\ \partial_x\varphi(t,-0) \end{array}\right)\, , \quad \forall t \in \mathbb{R}\, , \tag{2.3}$$

where

$$ad - bc = 1\, , \qquad a, ..., d \in \mathbb{R}\, . \tag{2.4}$$

We observe that a and d are dimensionless, whereas b and c have a non-trivial and opposite dimension.

The dual field $\widetilde{\varphi}(t,x)$ also satisfies

$$\left(\partial_t^2 - \partial_x^2\right) \widetilde{\varphi}(t,x) = 0\, , \qquad x \neq 0\, , \tag{2.5}$$

and as usual is related to $\varphi(t,x)$ by

$$\partial_t\widetilde{\varphi}(t,x) = -\partial_x\varphi(t,x)\, , \quad \partial_x\widetilde{\varphi}(t,x) = -\partial_t\varphi(t,x)\, , \qquad x \neq 0\, . \tag{2.6}$$

Eqs. (2.1-2.6) have a unique solution $\{\varphi\, ,\widetilde{\varphi}\}$, which represents the basis for bosonization with a point-like defect. In this paper we mostly concentrate on the case when impurity bound states are absent. This case is characterized [23] by the following additional constraints on the parameters:

$$\begin{cases} \frac{a+d+\sqrt{(a-d)^2+4}}{2b} \geq 0\, , & \text{for } b < 0\, , \\ \frac{c}{a+d} \geq 0\, , & \text{for } b = 0\, , \\ \frac{a+d-\sqrt{(a-d)^2+4}}{2b} \geq 0\, , & \text{for } b > 0\, . \end{cases} \tag{2.7}$$

In this domain the solution $\{\varphi\, ,\widetilde{\varphi}\}$ can be written in the form

$$\varphi(t,x) = \varphi_+(t,x) + \varphi_-(t,x)\, , \qquad \widetilde{\varphi}(t,x) = \widetilde{\varphi}_+(t,x) + \widetilde{\varphi}_-(t,x)\, , \tag{2.8}$$

where

$$\varphi_\pm(t,x) = \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk}{2\pi\sqrt{2|k|}} \left[a^{*\pm}(k) \mathrm{e}^{i|k|t-ikx} + a_\pm(k)\mathrm{e}^{-i|k|t+ikx}\right]\, , \tag{2.9}$$

$$\widetilde{\varphi}_\pm(t,x) = \theta(\pm x) \int_{-\infty}^{+\infty} \frac{dk\, \varepsilon(k)}{2\pi\sqrt{2|k|}} \left[a^{*\pm}(k) \mathrm{e}^{i|k|t - ikx} + a_\pm(k) \mathrm{e}^{-i|k|t+ikx}\right] . \tag{2.10}$$

These expressions have the familiar form of superpositions of creation $a^{*\pm}(k)$ and annihilation $a_\pm(k)$ operators. The interaction with the impurity deforms [11] only their commutation relations, which read now

$$a_{\xi_1}(k_1)\, a_{\xi_2}(k_2) - a_{\xi_2}(k_2)\, a_{\xi_1}(k_1) = 0\,, \tag{2.11}$$

$$a^{*\xi_1}(k_1)\, a^{*\xi_2}(k_2) - a^{*\xi_2}(k_2)\, a^{*\xi_1}(k_1) = 0\,, \tag{2.12}$$

$$\begin{gathered} a_{\xi_1}(k_1)\, a^{*\xi_2}(k_2) - a^{*\xi_2}(k_2)\, a_{\xi_1}(k_1) = \\ \left[\delta_{\xi_1}^{\xi_2} + T_{\xi_1}^{\xi_2}(k_1)\right] 2\pi\delta(k_1 - k_2)\, \mathbf{1} + R_{\xi_1}^{\xi_2}(k_1) 2\pi\delta(k_1+k_2)\, \mathbf{1}\,, \end{gathered} \tag{2.13}$$

where

$$R_+^+(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 + i(a+d)k - c}\,, \qquad R_-^-(k) = \frac{bk^2 + i(a-d)k + c}{bk^2 - i(a+d)k - c}\,, \tag{2.14}$$

$$T_+^-(k) = \frac{2ik}{bk^2 + i(a+d)k - c}\,, \qquad T_-^+(k) = \frac{-2ik}{bk^2 - i(a+d)k - c}\,, \tag{2.15}$$

are the *reflection* and *transmission coefficients* from the impurity. The associated *reflection* and *transmission matrices*

$$R(k) = \begin{pmatrix} R_+^+(k) & 0 \\ 0 & R_-^-(k) \end{pmatrix}\,, \qquad T(k) = \begin{pmatrix} 0 & T_+^-(k) \\ T_-^+(k) & 0 \end{pmatrix}\,, \tag{2.16}$$

satisfy hermitian analyticity

$$R(k)^\dagger = R(-k)\,, \qquad T(k)^\dagger = T(k)\,, \tag{2.17}$$

and unitarity

$$T(k)T(k) + R(k)R(-k) = \mathbb{I}\,, \tag{2.18}$$

$$T(k)R(k) + R(k)T(-k) = 0\,. \tag{2.19}$$

The exchange relations (2.11-2.13) deserve some comments. We observe first of all that (2.11-2.13) preserve the conventional initial conditions (2.2). In a slightly more general form the relations (2.11-2.13) appeared for the first time [11, 13] in the context of integrable models with impurities. The associative algebra generated by $\{a^{*\pm}(k),\, a_\pm(k),\, \mathbf{1}\}$, satisfying (2.11-2.13) and the constraints

$$a_\xi(k) = T_\xi^\eta(k) a_\eta(k) + R_\xi^\eta(k) a_\eta(-k)\,, \tag{2.20}$$

$$a^{*\xi}(k) = a^{*\eta}(k) T_\eta^\xi(k) + a^{*\eta}(-k) R_\eta^\xi(-k)\,, \tag{2.21}$$

has been called reflection-transmission (RT) algebra because it translates the analytic boundary conditions (2.3) in algebraic terms, directly related to the physical reflection and transmission amplitudes (2.14, 2.15). For this reason RT algebras represent a natural and universal tool for studying QFT with defects [16, 17, 23, 24] and it is not at all surprising that they appear also in the process of bosonization with impurities.

The derivation of the correlation functions of $\{\varphi\, , \widetilde{\varphi}\}$ in the Fock representation [13] of the RT algebra (2.11-2.13) is straightforward. It is convenient to change basis introducing the right and left chiral fields

$$\varphi_{_R}(t,x) = \varphi(t,x) + \widetilde{\varphi}(t,x)\, , \qquad \varphi_{_L}(t,x) = \varphi(t,x) - \widetilde{\varphi}(t,x)\, . \tag{2.22}$$

Inserting (2.8-2.10) in (2.22) one gets

$$\varphi_{_R}(t,x) = \theta(x)\varphi_{_{+R}}(t-x) + \theta(-x)\varphi_{_{-R}}(t-x)\, , \tag{2.23}$$
$$\varphi_{_L}(t,x) = \theta(x)\varphi_{_{+L}}(t+x) + \theta(-x)\varphi_{_{-L}}(t+x)\, , \tag{2.24}$$

where

$$\varphi_{\pm R}(\xi) = \int_0^{+\infty} \frac{dk}{\pi\sqrt{2k}} \left[a^{*\pm}(k)\mathrm{e}^{ik\xi} + a_\pm(k)\mathrm{e}^{-ik\xi}\right]\, , \tag{2.25}$$

$$\varphi_{\pm L}(\xi) = \int_0^{+\infty} \frac{dk}{\pi\sqrt{2k}} \left[a^{*\pm}(-k)\mathrm{e}^{ik\xi} + a_\pm(-k)\mathrm{e}^{-ik\xi}\right]\, . \tag{2.26}$$

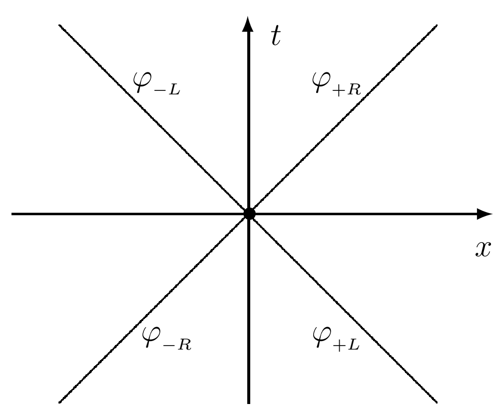

Fig. 1. The localization of $\varphi_{\pm R}$ and $\varphi_{\pm L}$ on the light cone.

The four components $\varphi_{\pm R}$ and $\varphi_{\pm L}$, whose localization is displayed on Fig. 1, couple each other through the defect at $x = 0$. This characteristic feature of our

system is captured by the correlation functions of $\varphi_{\pm R}$ and $\varphi_{\pm L}$, we are going to derive now. Using (2.13) and the fact that $a_\pm(k)$ annihilate the Fock vacuum, one gets the following two-point functions

$$\begin{aligned} \langle \varphi_{+R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \langle \varphi_{-R}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \\ \langle \varphi_{+L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \langle \varphi_{-L}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \\ = \int_0^{+\infty} \frac{dk}{\pi}(k^{-1})_{\mu_0} \mathrm{e}^{-ik\xi_{12}}\, , \qquad \xi_{12} \equiv \xi_1 - \xi_2\, , \end{aligned} \tag{2.27}$$

$(k^{-1})_{\mu_0}$ being the distribution [29]

$$(k^{-1})_{\mu_0} = \frac{d}{dk} \ln \frac{k}{\mu_0}\, . \tag{2.28}$$

The derivative here is understood in the sense of distributions and μ_0 is a free parameter with dimension of mass having a well-known (see e.g. [30]) infrared origin. We observe for further use that the identity

$$k\, (k^{-1})_{\mu_0} = 1 \tag{2.29}$$

holds on $\mathbb{R}_+$ and that

$$\int_0^{+\infty} \frac{dk}{\pi}(k^{-1})_{\mu_0} \mathrm{e}^{-ik\xi} = u(\mu\xi)\, , \qquad \mu \equiv \mu_0 \mathrm{e}^{\gamma_E}\, , \tag{2.30}$$

where

$$u(\xi) = -\frac{1}{\pi} \ln(|\xi|) - \frac{i}{2}\varepsilon(\xi) = -\frac{1}{\pi}\ln(i\xi + \epsilon)\, , \quad \epsilon > 0\, , \tag{2.31}$$

and γ_E is the Euler constant. The correlators (2.27) do not depend on the defect and coincide with the familiar defect-free ones. This conclusion obviously holds also for the commutators

$$[\varphi_{+R}(\xi_1)\, , \, \varphi_{+R}(\xi_2)] = [\varphi_{-R}(\xi_1)\, , \, \varphi_{-R}(\xi_2)] = -i\varepsilon(\xi_{12})\, , \tag{2.32}$$

$$[\varphi_{+L}(\xi_1)\, , \, \varphi_{+L}(\xi_2)] = [\varphi_{-L}(\xi_1)\, , \, \varphi_{-L}(\xi_2)] = -i\varepsilon(\xi_{12})\, , \tag{2.33}$$

which follow directly from eqs. (2.27,2.30,2.31).

The defect shows up in the mixed correlation functions in the following way. The transmission relates the plus and minus components with the same chirality:

$$\langle \varphi_{+R}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \int_0^{+\infty} \frac{dk}{\pi}(k^{-1})_{\mu_0} \mathrm{e}^{-ik\xi_{12}} T_+^-(k)\, , \tag{2.34}$$

$$\langle\varphi_{-R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}T_-^+(k)\,, \tag{2.35}$$

$$\langle\varphi_{+L}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}T_+^-(-k)\,, \tag{2.36}$$

$$\langle\varphi_{-L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}T_-^+(-k)\,. \tag{2.37}$$

The reflection instead relates different chiralities on the same half-line according to

$$\langle\varphi_{+R}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}R_+^+(k)\,, \tag{2.38}$$

$$\langle\varphi_{-R}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}R_-^-(k)\,, \tag{2.39}$$

$$\langle\varphi_{+L}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}R_+^+(-k)\,, \tag{2.40}$$

$$\langle\varphi_{-L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \int_0^{+\infty}\frac{dk}{\pi}(k^{-1})_{\mu_0}\mathrm{e}^{-ik\xi_{12}}R_-^-(-k)\,. \tag{2.41}$$

Finally,

$$\begin{aligned}\langle\varphi_{+R}(\xi_1)\varphi_{-L}(\xi_2)\rangle &= \langle\varphi_{-L}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \\ \langle\varphi_{-R}(\xi_1)\varphi_{+L}(\xi_2)\rangle &= \langle\varphi_{+L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = 0\,.\end{aligned} \tag{2.42}$$

The plus-minus and left-right mixing is captured by (2.34-2.37) and (2.38-2.41) respectively and is a direct consequence of the impurity. In a simpler form this phenomenon appears also in the case of boundary conformal field theory [31].

The integral representations (2.34-2.41) determine well-defined distributions allowing to analyze the locality properties of $\{\varphi,\, \widetilde{\varphi}\}$. The explicit form of the commutators at generic points t_1, x_1 and t_2, x_2 is quite involved. Fortunately however it drastically simplifies at space-like separated points. In fact, in the domain $t_{12}^2 - x_{12}^2 < 0$ one finds

$$[\varphi(t_1,x_1)\,,\,\varphi(t_2,x_2)] = [\widetilde{\varphi}(t_1,x_1)\,,\,\widetilde{\varphi}(t_2,x_2)] = 0\,, \tag{2.43}$$

$$[\varphi(t_1,x_1)\,,\,\widetilde{\varphi}(t_2,x_2)] = \frac{i}{2}[\varepsilon(x_{12}) + \varepsilon(\widetilde{x}_{12})]\theta(x_1x_2)\,, \tag{2.44}$$

where $\widetilde{x}_{12} \equiv x_1 + x_2$. Therefore, like in the case without defects, φ and $\widetilde{\varphi}$ are *local* fields, but *not relatively local*. As recognized already in the early sixties [27], this feature is the corner stone of bosonization. We will make essential use of it in the

next section, establishing the statistics of the vertex operators and constructing, in particular, fermions from bosons.

The symmetry properties of our system are strongly influenced by the impurity, which breaks down the 1+1 dimensional Poincaré group except of the invariance under time translations. Therefore the energy is conserved in accordance with the fact that our defects do not dissipate.

Concerning the internal symmetries, we introduce the charges

$$Q_{\epsilon Z} \equiv \int_{-\infty}^{+\infty} d\xi \, \partial_\xi \, \varphi_{\epsilon Z}(\xi) \, , \qquad \epsilon = \pm \, , \quad Z = R, \, L \, , \tag{2.45}$$

which are the building blocks for constructing the Klein factors [6] used in the vertex construction of affine Kac-Moody algebras. By definition $Q_{\epsilon Z}$ depend only on the asymptotic behavior of $\varphi_{\epsilon Z}(\xi)$ at $\xi = \pm\infty$. Using the correlation functions (2.27,2.34-2.41) one finds

$$[Q_{\epsilon_1 R} \, , \varphi_{\epsilon_2 R}(\xi)] = - \, [Q_{\epsilon_1 R} \, , \varphi_{\epsilon_2 L}(\xi)] = -i\delta_{\epsilon_1 \epsilon_2} \, , \tag{2.46}$$

$$[Q_{\epsilon_1 L} \, , \varphi_{\epsilon_2 R}(\xi)] = - \, [Q_{\epsilon_1 L} \, , \varphi_{\epsilon_2 L}(\xi)] = -i\delta_{\epsilon_1 \epsilon_2} \, . \tag{2.47}$$

Summarizing, the defect divides each left and right branches $C_{_R}$ and $C_{_L}$ of the light cone $C = C_{_R} \cup C_{_L}$ in two components $C_{\pm R}$ and $C_{\pm L}$, where the chiral fields $\varphi_{\pm R}$ and $\varphi_{\pm L}$ are localized. These fields are not independent: the reflection and transmission coefficients define a specific interaction between the left-right and plus-minus components respectively. In the next subsection we describe two concrete sets of parameters $\{a, \, b, \, c, \, d\}$, which nicely illustrate both the above general structure and the characteristic features of bosonization with defects.

2.1 Examples:

- **quasi-conformal defects;**

We start by considering the one-parameter family of defects

$$\{a = 1/\lambda, \, 0, \, 0, \, d = \lambda \neq 0\} \, . \tag{2.48}$$

Since the dimensional parameters b and c are set to 0, we call them quasi-conformal defects. These defects coincide with the permeable conformal walls introduced in [25, 26]. From (2.14,2.15) one gets

$$R_+^+(k) = -R_-^-(k) = r(\lambda) \equiv \frac{1-\lambda^2}{1+\lambda^2} \, , \qquad T_+^-(k) = T_-^+(k) = 1 - r(\lambda) \, . \tag{2.49}$$

Accordingly, one has in addition to (2.27) the following non-trivial correlation functions

$$\begin{aligned}
\langle \varphi_{+R}(\xi_1)\varphi_{-R}(\xi_2)\rangle &= \langle \varphi_{-R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \\
\langle \varphi_{+L}(\xi_1)\varphi_{-L}(\xi_2)\rangle &= \langle \varphi_{-L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = [1-r(\lambda)]\, u(\mu\xi_{12})\, ,
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
\langle \varphi_{+R}(\xi_1)\varphi_{+L}(\xi_2)\rangle &= -\langle \varphi_{-R}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \\
\langle \varphi_{+L}(\xi_1)\varphi_{+R}(\xi_2)\rangle &= -\langle \varphi_{-L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = r(\lambda)\, u(\mu\xi_{12})\, ,
\end{aligned} \tag{2.51}$$

which vanish in the conformal case. All correlators (2.27,2.30,2.31,2.50,2.51) of the quasi-conformal defect are expressed in terms of the logarithm $u(\mu\xi)$ and the parameter λ. In addition to the universal (defect independent) commutators (2.32,2.33) one has:

$$[\varphi_{+R}(\xi_1)\, ,\, \varphi_{-R}(\xi_2)] = [\varphi_{-L}(\xi_1)\, ,\, \varphi_{+L}(\xi_2)] = -i[1-r(\lambda)]\, \varepsilon(\xi_{12})\, , \tag{2.52}$$

$$[\varphi_{+R}(\xi_1)\, ,\, \varphi_{+L}(\xi_2)] = -[\varphi_{-L}(\xi_1)\, ,\, \varphi_{-R}(\xi_2)] = -i\, r(\lambda)\, \varepsilon(\xi_{12})\, . \tag{2.53}$$

- **δ-defects;**

As a second example we consider the impurities defined by

$$\{a = d = 1,\, b = 0,\, c = 2\eta > 0\}\, . \tag{2.54}$$

One usually refers to this one-parameter family as δ-defects, because they can be implemented by coupling φ to the external potential $U(x) = 2\eta\delta(x)$. The reflection and transmission coefficients take the form

$$R_+^+(k) = \frac{-i\eta}{k+i\eta}\, , \qquad R_-^-(k) = \frac{i\eta}{k-i\eta}\, , \tag{2.55}$$

$$T_+^-(k) = \frac{k}{k+i\eta}\, , \qquad T_-^+(k) = \frac{k}{k-i\eta}\, . \tag{2.56}$$

It is worth mentioning that

$$\lim_{k\to\infty} R_+^+(k) = \lim_{k\to\infty} R_-^-(k) = 0\, , \tag{2.57}$$

which is actually an exclusive feature of the δ-defects. In fact, it follows from (2.14,2.15) that (2.55,2.56) define the most general defects satisfying (2.57), which

implies in turn that the correlation functions (2.38-2.41) are regular for $\xi_1 = \xi_2$. Indeed, inserting (2.55,2.56) in (2.34-2.41) and performing the integration over k one gets the following explicit two-point functions

$$\langle \varphi_{+R}(\xi_1)\varphi_{-R}(\xi_2)\rangle = \langle \varphi_{-L}(\xi_1)\varphi_{+L}(\xi_2)\rangle = v_-(\eta\xi_{12})\,, \tag{2.58}$$

$$\langle \varphi_{-R}(\xi_1)\varphi_{+R}(\xi_2)\rangle = \langle \varphi_{+L}(\xi_1)\varphi_{-L}(\xi_2)\rangle = v_+(-\eta\xi_{12})\,, \tag{2.59}$$

$$\langle \varphi_{+R}(\xi_1)\varphi_{+L}(\xi_2)\rangle = \langle \varphi_{-L}(\xi_1)\varphi_{-R}(\xi_2)\rangle = v_-(\eta\xi_{12}) - u(\mu\xi_{12})\,, \tag{2.60}$$

$$\langle \varphi_{-R}(\xi_1)\varphi_{-L}(\xi_2)\rangle = \langle \varphi_{+L}(\xi_1)\varphi_{+R}(\xi_2)\rangle = v_+(-\eta\xi_{12}) - u(\mu\xi_{12})\,, \tag{2.61}$$

where u is defined by (2.31) and

$$v_\pm(\xi) \equiv -\frac{1}{\pi}\mathrm{e}^{-\xi}\,\mathrm{Ei}(\xi \pm \mathrm{i}\epsilon)\,, \qquad \epsilon > 0\,, \tag{2.62}$$

Ei being the exponential-integral function. Recalling the expansion

$$\mathrm{Ei}(\xi \pm \mathrm{i}\epsilon) = \gamma_{\mathrm{E}} + \ln(\xi \pm \mathrm{i}\epsilon) + \sum_{\mathrm{n}=1}^{\infty}\frac{\xi^{\mathrm{n}}}{\mathrm{n}\cdot\mathrm{n}!}\,, \tag{2.63}$$

we see that $u(\xi)$ and $v_\pm(\mp\xi)$ have the same logarithmic singularity in $\xi = 0$, confirming that the correlators (2.60,2.61) are not singular at $\xi_1 = \xi_2$.

From (2.58-2.61) one gets the commutators:

$$[\varphi_{+R}(\xi_1)\,,\,\varphi_{-R}(\xi_2)] = [\varphi_{-L}(\xi_1)\,,\,\varphi_{+L}(\xi_2)] = -2i\theta(\xi_{12})\mathrm{e}^{-\eta\xi_{12}}\,, \tag{2.64}$$

$$[\varphi_{+R}(\xi_1)\,,\,\varphi_{+L}(\xi_2)] = [\varphi_{-L}(\xi_1)\,,\,\varphi_{-R}(\xi_2)] = -2i\theta(\xi_{12})\mathrm{e}^{-\eta\xi_{12}} + i\varepsilon(\xi_{12})\,. \tag{2.65}$$

We stress that in deriving (2.58-2.61) we essentially used that $\eta > 0$. The correlators (2.58-2.61) are singular in the limit $\eta \to 0$, which forbids to recover from them the free case $\eta = 0$. Such type of discontinuity appears [29] also on the half-line between the scalar field quantized with Robin and Neumann boundary conditions.

3 Vertex operators in presence of defects

We have enough background at this point for constructing vertex operators. For any couple $\zeta \equiv (\alpha, \beta) \in \mathbb{R}^2$ we introduce the field

$$V(t, x; \zeta) =: \exp[i\sqrt{\pi}(\alpha\varphi + \beta\widetilde{\varphi})] : (t, x)\,, \tag{3.1}$$

where the normal ordering : : is taken with respect to the creation and annihilation operators $\{a^{*\pm}(k),\, a_\pm(k)\}$. Like in the case without defect, the operators (3.1) generate an algebra $\mathcal{V}$. The exchange properties of the vertex operators $V(t,x;\zeta)$ determine their statistics. A standard calculation shows that

$$V(t_1,x_1;\zeta_1)V(t_2,x_2;\zeta_2) = \mathcal{E}(t_{12},x_1,x_2;\zeta_1,\zeta_2)\, V(t_2,x_2;\zeta_2)V(t_1,x_1;\zeta_1)\,, \tag{3.2}$$

the exchange factor $\mathcal{E}$ being

$$\mathcal{E}(t_{12},x_1,x_2;\zeta_1,\zeta_2) = \mathrm{e}^{-\pi[\alpha_1\varphi(t_1,x_1)+\beta_1\widetilde{\varphi}(t_1,x_1)\,,\,\alpha_2\varphi(t_2,x_2)+\beta_2\widetilde{\varphi}(t_2,x_2)]}\,. \tag{3.3}$$

The statistics of $V(t,x;\zeta)$ is determined by the value of (3.3) at space-like distances $t_{12}^2 - x_{12}^2 < 0$. By means of (2.43,2.44) one finds in this domain

$$\mathcal{E}(t_{12},x_1,x_2;\zeta_1,\zeta_2) = \mathrm{e}^{\frac{i\pi}{2}[(\alpha_1\beta_2+\alpha_2\beta_1)\varepsilon(x_{12})+(\alpha_1\beta_2-\alpha_2\beta_1)\varepsilon(\widetilde{x}_{12})]\theta(x_1x_2)}\,. \tag{3.4}$$

Setting $\zeta_1=\zeta_2\equiv\zeta$ in (3.4) one obtains

$$\mathcal{E}(t_{12},x_1,x_2;\zeta,\zeta) = \mathrm{e}^{i\pi\alpha\beta\varepsilon(x_{12})\theta(x_1x_2)}\,, \tag{3.5}$$

which governs the statistics of $V(t,x;\zeta)$.

It follows from (3.5) that the exchange properties of the vertex operators depend not only on the parameters $(\alpha,\,\beta)$, but also on the position. This is a new phenomenon in the context of bosonization, which has its origin in the breakdown of translation invariance by the impurity. The θ-factor in the exponent of (3.5) implies that two vertex operators localized on the opposite sides of the impurity are exchanged as bosons. However, when the vertex operators are localized on the same half-line, they behave as anyons with statistics parameter

$$\vartheta \equiv \alpha\beta\,. \tag{3.6}$$

For $\vartheta = 2k$ and $\vartheta = 2k+1$ with $k\in\mathbb{Z}$ one recovers Bose and Fermi statistics respectively. The remaining values of ϑ lead to abelian braid (anyon) statistics.

It is instructive to consider the vertex algebras $\mathcal{V}_{\pm R}$ and $\mathcal{V}_{\pm L}$ generated by

$$V(t,x\gtrless 0;(\alpha,\alpha)) =: \exp[i\sqrt{\pi}\alpha\varphi_{\pm R}] : (t-x) \equiv V_{\pm R}(t-x;\alpha)\,, \tag{3.7}$$

$$V(t,x\gtrless 0;(\alpha,-\alpha)) =: \exp[i\sqrt{\pi}\alpha\varphi_{\pm L}] : (t+x) \equiv V_{\pm L}(t+x;\alpha)\,, \tag{3.8}$$

respectively. These vertex operators are localized on the branches $C_{\pm R}$ and $C_{\pm L}$ of the light cone. Up to unessential μ-dependent multiplicative factor the universal (defect independent) vertex correlators are

$$\begin{aligned}\langle V_{+R}(\xi_1;\alpha)V^*_{+R}(\xi_2;\alpha)\rangle &= \langle V_{+R}(\xi_1;\alpha)V^*_{+R}(\xi_2;\alpha)\rangle\\ \langle V_{+L}(\xi_1;\alpha)V^*_{+L}(\xi_2;\alpha)\rangle &= \langle V_{+L}(\xi_1;\alpha)V^*_{+L}(\xi_2;\alpha)\rangle \sim (\xi_{12}-i\epsilon)^{-\alpha^2}\,.\end{aligned} \tag{3.9}$$

As expected, (3.9) is a homogeneous function of degree $-\alpha^2$.

Due to the interaction with the defect, there exist also non-trivial mixed vertex correlation functions, depending on the parameters $\{a,\, b,\, c,\, d\}$. The quasi-conformal defects (2.48) lead for instance to

$$\begin{aligned} \langle V_{+R}(\xi_1;\alpha)V^*_{-R}(\xi_2;\alpha)\rangle &= \langle V_{-R}(\xi_1;\alpha)V^*_{+R}(\xi_2;\alpha)\rangle = \\ \langle V_{+L}(\xi_1;\alpha)V^*_{-L}(\xi_2;\alpha)\rangle &= \langle V_{-L}(\xi_1;\alpha)V^*_{+L}(\xi_2;\alpha)\rangle \sim (\xi_{12}-i\epsilon)^{-\frac{2\lambda^2}{1+\lambda^2}\alpha^2}\, , \end{aligned} \tag{3.10}$$

$$\begin{aligned} \langle V_{+R}(\xi_1;\alpha)V^*_{+L}(\xi_2;\alpha)\rangle &= \langle V_{-R}(\xi_1;\alpha)V^*_{-L}(\xi_2;\alpha)\rangle = \\ \langle V_{+L}(\xi_1;\alpha)V^*_{+R}(\xi_2;\alpha)\rangle &= \langle V_{-L}(\xi_1;\alpha)V^*_{-R}(\xi_2;\alpha)\rangle \sim (\xi_{12}-i\epsilon)^{-\frac{1-\lambda^2}{1+\lambda^2}\alpha^2}\, . \end{aligned} \tag{3.11}$$

One has still homogeneous functions, whose degree is however λ-dependent. For impurities involving non-vanishing dimensional parameters b and/or c the correlation functions (3.10,3.11) are no longer homogeneous functions of ξ_{12}. For the δ-defect (2.54) one gets for instance

$$\langle V_{+R}(\xi_1;\alpha)V^*_{-R}(\xi_2;\alpha)\rangle = \langle V_{-R}(\xi_1;\alpha)V^*_{+R}(\xi_2;\alpha)\rangle = \mathrm{e}^{\pi\alpha^2 v_-(\eta\xi_{12})}\, , \tag{3.12}$$
$$\langle V_{+L}(\xi_1;\alpha)V^*_{-L}(\xi_2;\alpha)\rangle = \langle V_{-L}(\xi_1;\alpha)V^*_{+L}(\xi_2;\alpha)\rangle = \mathrm{e}^{\pi\alpha^2 v_+(-\eta\xi_{12})}\, , \tag{3.13}$$

$$\langle V_{+R}(\xi_1;\alpha)V^*_{+L}(\xi_2;\alpha)\rangle = \langle V_{-R}(\xi_1;\alpha)V^*_{-L}(\xi_2;\alpha)\rangle = \mathrm{e}^{\pi\alpha^2[v_-(\eta\xi_{12})-u(\mu\xi_{12})]}\, , \tag{3.14}$$
$$\langle V_{+L}(\xi_1;\alpha)V^*_{+R}(\xi_2;\alpha)\rangle = \langle V_{-L}(\xi_1;\alpha)V^*_{-R}(\xi_2;\alpha)\rangle = \mathrm{e}^{\pi\alpha^2[v_+(-\eta\xi_{12})-u(\mu\xi_{12})]}, \tag{3.15}$$

which contain the exponential-integral function (2.63) and are not homogeneous.

As a first application of the vertex algebra $\mathcal{V}$ for generic defect $\{a,\, b,\, c,\, d\}$ we consider the bosonization of the free massless Dirac field with impurity. In order to fix the notation, we first recall the massless Dirac equation on $\mathbb{R}\setminus\{0\}$. One has

$$(\gamma_t\partial_t - \gamma_x\partial_x)\psi(t,x) = 0\, , \qquad x\in\mathbb{R}\setminus\{0\}\, , \tag{3.16}$$

where

$$\psi(t,x) = \begin{pmatrix}\psi_1(t,x)\\ \psi_2(t,x)\end{pmatrix}, \qquad \gamma_t = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \gamma_x = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{3.17}$$

The standard vector and axial currents are

$$j_\nu(t,x) = \overline{\psi}(t,x)\gamma_\nu\psi(t,x)\, , \qquad j^5_\nu(t,x) = \overline{\psi}(t,x)\gamma_\nu\gamma^5\psi(t,x)\, , \qquad \nu = t,x\, , \tag{3.18}$$

with $\overline{\psi} \equiv \psi^* \gamma_t$ and $\gamma^5 \equiv -\gamma_t \gamma_x$. From eq.(3.16) it follows that both j_ν and j^5_ν are conserved. Moreover, the γ^5-identities $\gamma_t \gamma^5 = -\gamma_x$ and $\gamma_x \gamma^5 = -\gamma_t$ imply the relations

$$j^5_t = -j_x \, , \qquad j^5_x = -j_t \, . \tag{3.19}$$

Our main goal now is to quantize (3.16) in terms of the fields φ and $\widetilde{\varphi}$, establishing the defect boundary conditions on ψ encoded in (2.3). For this purpose we set

$$\psi_1(t,x) = \mu^{\alpha^2/2} V(t,x;\zeta_1 = (\alpha,\alpha)) = \mu^{\alpha^2/2} :\exp(i\sqrt{\pi}\alpha\varphi_{_R}):(t,x)\, , \tag{3.20}$$

$$\psi_2(t,x) = \mu^{\alpha^2/2} V(t,x;\zeta_2 = (-\alpha,\alpha)) = \mu^{\alpha^2/2} :\exp(i\sqrt{\pi}\alpha\varphi_{_L}):(t,x)\, , \tag{3.21}$$

where μ is the infrared scale introduced in (2.30).

One easily verifies that ψ, defined by eqs. (3.20,3.21), satisfies the Dirac equation on $\mathbb{R} \setminus \{0\}$. Moreover, one has the anticommutation relations

$$\psi_i(t_1,x_1)\psi_j(t_2,x_2) = -\psi_j(t_2,x_2)\psi_i(t_1,x_1)\, , \tag{3.22}$$

for $|t_{12}| < |x_1 - x_2|$ and $x_1 x_2 > 0$, provided that

$$\alpha^2 = 2k+1\, , \qquad k \in \mathbb{N}\, , \tag{3.23}$$

which is imposed to the end of this section.

The next step is to construct the quantum currents (3.18). We adopt the point-splitting procedure, considering the limits

$$j_\nu(t,x) = \frac{1}{2}\lim_{\sigma\to\pm 0} Z(\sigma)\left[\,\overline{\psi}(t,x)\gamma_\nu\psi(t,x+\sigma) + \overline{\psi}(t,x+\sigma)\gamma_\nu\psi(t,x)\,\right]\, , \qquad x \gtrless 0\, , \tag{3.24}$$

where $Z(\sigma)$ implements the renormalization. The basic general formula for evaluating (3.24), is obtained by normal ordering the product $V^*(t,x+\sigma;\zeta)V(t,x;\zeta)$. One has

$$\begin{aligned} V^*(t,x+\sigma;\zeta)V(t,x;\zeta) \ &= \\ :\exp\Big\{i\sqrt{\pi}[\alpha\varphi(t,x) - \alpha\varphi(t,x+\sigma) \ &+ \ \beta\widetilde{\varphi}(t,x) - \beta\widetilde{\varphi}(t,x+\sigma)]\Big\}: \\ \exp\Big\{\frac{\pi}{4}[(\alpha+\beta)^2\langle\varphi_{_R}(t,x+\sigma)\varphi_{_R}(t,x)\rangle \ &+ \ (\alpha-\beta)^2\langle\varphi_{_L}(t,x+\sigma)\varphi_{_L}(t,x)\rangle \\ +(\alpha^2-\beta^2)(\langle\varphi_{_R}(t,x+\sigma)\varphi_{_L}(t,x)\rangle \ &+ \ \langle\varphi_{_L}(t,x+\sigma)\varphi_{_R}(t,x)\rangle)]\Big\}\, . \end{aligned} \tag{3.25}$$

For the special values $\zeta_1 = (\alpha,\,\alpha)$ and $\zeta_2 = (-\alpha,\,\alpha)$ the mixed $R-L$ and $L-R$ correlation functions drop out. Setting

$$Z(\sigma) = \frac{-\sigma^{\alpha^2-1}}{2\alpha\sin\left(\frac{\pi}{2}\alpha^2\right)}\, , \tag{3.26}$$

and performing the limit in (3.24), one finds the conserved current

$$j_\nu(t,x) = \sqrt{\pi}\,\partial_\nu\varphi(t,x)\,. \tag{3.27}$$

Thus one recovers the same type of relation as in conventional bosonization [27] without impurities.

In analogy with (3.24) we introduce the axial current by

$$j^5_\nu(t,x) = \frac{1}{2}\lim_{\sigma\to\pm 0} Z(\sigma)\left[\,\overline{\psi}(t,x)\gamma_\nu\gamma^5\psi(t,x+\sigma) + \overline{\psi}(t,x+\sigma)\gamma_\nu\gamma^5\psi(t,x)\,\right]\,,\quad x \gtrless 0\,, \tag{3.28}$$

The vector current result and the γ^5-identities directly imply that the limit in the right hand side of (3.28) exists and

$$j^5_\nu(t,x) = \sqrt{\pi}\,\partial_\nu\widetilde{\varphi}(t,x)\,. \tag{3.29}$$

The classical relations (3.19) are thus respected on quantum level as well.

Eqs.(3.27,3.29) imply that the defect boundary conditions on ψ at $x=0$ are most conveniently formulated in terms of the currents, which are the simplest observables of the fermion field. Combining (2.3) with (3.27,3.29) one obtains

$$\int_{+0}^{+\infty} dx\, j_x(t,x) = a\int_{-\infty}^{-0} dx\, j_x(t,x) + b\, j_x(t,-0)\,, \tag{3.30}$$

$$j_x(t,+0) = c\int_{-\infty}^{-0} dx\, j_x(t,x) + d\, j_x(t,-0)\,. \tag{3.31}$$

The linear and local boundary conditions on φ are therefore translated in both non-linear and non-local conditions on ψ.

Summarizing, we have shown that vertex operators in the presence of a point-like defect admit x-dependent anyon statistics. Afterwards we established a bosonization procedure for the free massless Dirac field in $\mathbb{R}\setminus 0$. The relative vector and axial currents have been expressed in terms of φ and $\widetilde{\varphi}$ respectively. Taking as an example the massless Thirring model with defect, we will extend in the next section the bosonization procedure to the case of current-current interactions.

4 Thirring model with defect

We will first solve the massless Thirring model [2] with a δ-defect (2.54), generalizing afterwards the solution to a generic point-like defect $\{a, b, c, d\}$. The classical dynamics of the model is governed by the equation of motion

$$i(\gamma_t\partial_t - \gamma_x\partial_x)\Psi(t,x) = g\left[\gamma_t J_t(t,x) - \gamma_x J_x(t,x)\right]\Psi(t,x)\,,\quad x\neq 0\,, \tag{4.1}$$

where $g \in \mathbb{R}$ is the coupling constant and J_ν is the conserved current

$$J_\nu(t,x) = \overline{\Psi}(t,x)\gamma_\nu \Psi(t,x)\,, \tag{4.2}$$

which, according to (3.30,3.31), satisfies the defect boundary conditions

$$\int_{+0}^{+\infty} dx\, J_x(t,x) \;=\; \int_{-\infty}^{-0} dx\, J_x(t,x)\,, \tag{4.3}$$

$$J_x(t,+0) - J_x(t,-0) \;=\; 2\eta \int_{-\infty}^{-0} dx\, J_x(t,x)\,. \tag{4.4}$$

For quantizing the system (4.1-4.4), we introduce the fields

$$\Psi_1(t,x) = \mu^\gamma\, V(t,x;\zeta_1 = (\alpha,\beta)) = \mu^\gamma : \exp[i\sqrt{\pi}(\alpha\varphi + \beta\widetilde{\varphi})] : (t,x)\,, \tag{4.5}$$

$$\Psi_2(t,x) = \mu^\gamma\, V(t,x;\zeta_2 = (\alpha,-\beta)) = \mu^\gamma : \exp[i\sqrt{\pi}(\alpha\varphi - \beta\widetilde{\varphi})] : (t,x)\,, \tag{4.6}$$

where

$$\gamma \equiv \frac{\alpha^2+\beta^2}{2} > 0\,, \tag{4.7}$$

and the correlation functions of $\{\varphi,\, \widetilde{\varphi}\}$ are fixed by (2.58-2.61). Moreover, we require

$$\alpha\beta = 2k+1\,, \qquad k \in \mathbb{Z}\,, \tag{4.8}$$

which according to the results of the previous section ensures Fermi statistics when $\Psi_{1,2}$ are localized at the same side of the defect.

The quantum current J_ν is constructed in analogy with (3.24), setting

$$J_\nu(t,x) = \frac{1}{2}\lim_{\sigma\to\pm 0} Z_\nu(x;\sigma)\left[\,\overline{\Psi}(t,x)\gamma_\nu\Psi(t,x+\sigma) + \overline{\Psi}(t,x+\sigma)\gamma_\nu\Psi(t,x)\,\right]\,, \quad x \gtrless 0\,, \tag{4.9}$$

without summation over ν. The presence of two x-dependent renormalization constants $Z_t(x;\sigma)$ and $Z_x(x;\sigma)$ in (4.9) is a consequence of the fact that both translation and Lorentz invariance are broken by the defect. In order to satisfy the defect boundary conditions (4.3,4.4), we shall determine $Z_t(x;\sigma)$ and $Z_x(x;\sigma)$ in such a way that

$$J_\nu(t,x) = \sqrt{\pi}\,\partial_\nu\varphi(t,x)\,. \tag{4.10}$$

For this purpose we first observe that the operator products under the limit (4.9) have the following expansions for $\sigma \to 0$:

$$\frac{1}{2}\left[\Psi_1^*(t,x+\sigma)\Psi_1(t,x) + \Psi_1^*(t,x)\Psi_1(t,x+\sigma)\right] =$$
$$\sigma^{1-\gamma}\left[\alpha\partial_x\varphi(t,x) + \beta\partial_x\widetilde{\varphi}(t,x) + O(\sigma^2)\right]\chi(x;\alpha,\beta)\,, \tag{4.11}$$

$$\frac{1}{2}\left[\Psi_2^*(t,x+\sigma)\Psi_2(t,x)+\Psi_2^*(t,x)\Psi_2(t,x+\sigma)\right]=$$
$$\sigma^{1-\gamma}\left[\alpha\partial_x\varphi(t,x)-\beta\partial_x\widetilde{\varphi}(t,x)+O(\sigma^2)\right]\ \chi(x;\alpha,\beta)\,, \tag{4.12}$$

where $\chi(x;\alpha,\beta)$ is a function given by

$$\chi(x;\alpha,\beta)=\sqrt{\pi}\sin\left(\frac{\pi}{2}\alpha\beta\right)\mathrm{e}^{\left\{\frac{\pi}{4}(\alpha^2-\beta^2)[v_+(-2\eta x)-u(2\mu x)+v_-(-2\eta x)-u(-2\mu x)]\right\}}\,. \tag{4.13}$$

We conclude therefore that defining

$$Z_t(x;\sigma)=\frac{-\sigma^{\gamma-1}\sqrt{\pi}}{2\beta\chi(x;\alpha,\beta)}\,,\qquad Z_x(x;\sigma)=\frac{-\sigma^{\gamma-1}\sqrt{\pi}}{2\alpha\chi(x;\alpha,\beta)}\,, \tag{4.14}$$

the limit (4.9) precisely reproduces (4.10).

Let us turn now to the quantum version of the equation of motion. In view of (4.10), one gets from (4.1)

$$i(\gamma_t\partial_t-\gamma_x\partial_x)\Psi(t,x)=g\sqrt{\pi}:(\gamma_t\partial_t\varphi-\gamma_x\partial_x\varphi)\,\Psi:(t,x)\,. \tag{4.15}$$

Now, using the explicit form (4.5,4.6) of Ψ, one easily verifies that (4.15) is satisfied provided that

$$\alpha-\beta=-g\,. \tag{4.16}$$

Combining eq. (4.8) and eq. (4.16), we obtain two families of solutions

$$\alpha_{1,2}=-\frac{g}{2}\pm\sqrt{\frac{g^2}{4}+(2k+1)}\,,\qquad \beta_{1,2}=\alpha_{1,2}+g\,, \tag{4.17}$$

parameterized by $k\in\mathbb{Z}$ with the constraint

$$2k+1\geq-\frac{g^2}{4}\,, \tag{4.18}$$

ensuring that $\alpha,\beta\in\mathbb{R}$. The freedom associated with $k\in\mathbb{Z}$ is present also in the Thirring model without defect. Since Lorentz invariance is preserved there, it is natural to require in addition that the Lorentz spin of Ψ takes the canonical value $\frac{1}{2}$, which fixes $k=0$.

The above solution of the Thirring model has a straightforward generalization to a generic defect $\{a,\,b,\,c,\,d\}$. In that case the function $\chi(x;\alpha,\beta)$ takes the form

$$\chi(x;\alpha,\beta)=\sqrt{\pi}\sin\left(\frac{\pi}{2}\alpha\beta\right)\mathrm{e}^{\left\{\frac{\pi}{4}(\alpha^2-\beta^2)\left[\langle\varphi_{+L}(x)\varphi_{+R}(-x)\rangle+\langle\varphi_{+R}(-x)\varphi_{+L}(x)\rangle\right]\right\}}\,. \tag{4.19}$$

It is easily seen that the corresponding change in the renormalization constants Z_ν does not affect the values (4.17) of the parameters α and β as functions of the coupling constant g.

5 Non-abelian bosonization

Following the Frenkel-Kac construction [6] of the vertex representation of the affine Kac-Moody algebra $\widehat{sl}(2)$, we introduce the operators

$$H_{\epsilon Z}(\xi) = \sqrt{\pi}\partial_\xi \varphi_{\epsilon Z}(\xi)\,, \qquad E^{\pm}_{\epsilon Z}(\xi) = \mu : \mathrm{e}^{\pm i\sqrt{2\pi}\varphi_{\epsilon Z}(\xi)} :, \tag{5.1}$$

where $\epsilon = \pm$, $Z = L, R$. Using eqs. (2.32,2.33), *for fixed* $\{\epsilon,\, Z\}$ one gets the well-known $\widehat{sl}(2)$ commutation relations:

$$[H_{\epsilon Z}(\xi_1)\,,\, H_{\epsilon Z}(\xi_2)] = 2\pi i\, \delta'(\xi_{12})\mathbb{I}\,, \tag{5.2}$$

$$[H_{\epsilon Z}(\xi_1)\,,\, E^{\pm}_{\epsilon Z}(\xi_2)] = \pm 2\pi\, \delta(\xi_{12})\sqrt{2}\, E^{\pm}_{\epsilon Z}(\xi_2)\,, \tag{5.3}$$

$$[E^{+}_{\epsilon Z}(\xi_1)\,,\, E^{-}_{\epsilon Z}(\xi_2)] = 2\pi i\, \delta'(\xi_{12})\mathbb{I} + 2\pi\, \delta(\xi_{12})\, H_{\epsilon Z}(\xi_1)\,, \tag{5.4}$$

$$[E^{+}_{\epsilon Z}(\xi_1)\,,\, E^{+}_{\epsilon Z}(\xi_2)] = [E^{-}_{\epsilon Z}(\xi_1)\,,\, E^{-}_{\epsilon Z}(\xi_2)] = 0\,. \tag{5.5}$$

In this way one recovers four vertex representations $\{\varrho_{\epsilon Z}\, :\, \epsilon = \pm,\, Z = R,\, L\}$ of $\widehat{sl}(2)$. This is not surprising because, as explained in section 2, the defect remains hidden when the theory is restricted on any of the components $C_{\epsilon Z}$ of the light cone. Keeping in mind that all four representations act in the Hilbert space where the vertex algebra $\mathcal{V}$ is represented, one can study also the interplay between the generators of different $\varrho_{\epsilon Z}$. Let us observe first of all that ϱ_{+R} and ϱ_{-L} as well as ϱ_{-R} and ϱ_{+L} commute because of (2.42). However, since $\{\varphi_{\epsilon Z}\}$ interact among themselves through the defect, there is a non-trivial interplay among the other four pairs $\{\varrho_{+R},\, \varrho_{+L}\}$, $\{\varrho_{-R},\, \varrho_{-L}\}$, $\{\varrho_{+R},\, \varrho_{-R}\}$ and $\{\varrho_{+L},\, \varrho_{-L}\}$ of representations. For a generic defect $\{a,\, b,\, c,\, d\}$ the commutator of two generators belonging to $\varrho_{\epsilon_1 Z_1}$ and $\varrho_{\epsilon_2 Z_2}$ is in general a *bilocal* operator of the type

$$B^{\,\pm\quad\ \pm}_{\epsilon_1 Z_1,\, \epsilon_2 Z_2}(\xi_1, \xi_2) \equiv\, : \mathrm{e}^{\pm i\sqrt{2\pi}\varphi_{\epsilon_1 Z_1}(\xi_1) \pm i\sqrt{2\pi}\varphi_{\epsilon_2 Z_2}(\xi_2)} : \,. \tag{5.6}$$

It turns out that the mixed commutators within the pairs $\{\varrho_{+R},\, \varrho_{+L}\}$, $\{\varrho_{-R},\, \varrho_{-L}\}$, $\{\varrho_{+R},\, \varrho_{-R}\}$ and $\{\varrho_{+L},\, \varrho_{-L}\}$ have all the same structure. So, let us consider for illustration the commutators between ϱ_{+R} and ϱ_{+L}. One finds

$$[H_{+R}(\xi_1)\,,\, H_{+L}(\xi_2)] = i\partial_{\xi_1} f(\xi_{12})\mathbb{I}\,, \tag{5.7}$$

$$[H_{+R}(\xi_1)\,,\, E^{\pm}_{+L}(\xi_2)] = \pm f(\xi_{12})\sqrt{2}\, E^{\pm}_{+L}(\xi_2)\,, \tag{5.8}$$

$$[H_{+L}(\xi_1)\,,\, E^{\pm}_{+R}(\xi_2)] = \mp f(-\xi_{12})\sqrt{2}\, E^{\pm}_{+R}(\xi_2)\,, \tag{5.9}$$

$$[E^{+}_{+R}(\xi_1)\,,\, E^{+}_{+L}(\xi_2)] = g_+(\xi_{12}) B^{\,+\quad +}_{+R,\, +L}(\xi_1, \xi_2)\,, \tag{5.10}$$

$$[E^-_{+R}(\xi_1)\,,\,E^-_{+L}(\xi_2)] = g_+(\xi_{12})B^{-\ \ -}_{+R,\,+L}(\xi_1,\xi_2)\,, \tag{5.11}$$

$$[E^+_{+R}(\xi_1)\,,\,E^-_{+L}(\xi_2)] = g_-(\xi_{12})B^{+\ \ -}_{+R,\,+L}(\xi_1,\xi_2)\,, \tag{5.12}$$

$$[E^-_{+R}(\xi_1)\,,\,E^+_{+L}(\xi_2)] = g_-(\xi_{12})B^{-\ \ +}_{+R,\,+L}(\xi_1,\xi_2)\,, \tag{5.13}$$

where f and $g_\pm$ are some functions depending on the defect and thus on the parameters $\{a,\, b,\, c,\, d\}$. For the quasi-conformal defects (2.48) one has

$$f(\xi) = 2\pi r(\lambda)\delta(\xi)\,, \qquad r(\lambda) = \frac{1-\lambda^2}{1+\lambda^2}\,, \tag{5.14}$$

$$g_\pm(\xi) = \pm 2i\,\mu^{2\pm r(\lambda)}\,\sin[\pi r(\lambda)]\,\varepsilon(\xi)\,|\xi|^{\pm 2r(\lambda)}\,. \tag{5.15}$$

The δ-defects (2.54) lead instead to

$$f(\xi) = -2\pi\,\eta\,\theta(\xi)\mathrm{e}^{-\eta\xi}\,, \tag{5.16}$$

$$g_\pm(\xi) = \pm 2i\,\mu^2\,\sin\left(2\pi\mathrm{e}^{-\eta\xi}\right)\,\theta(\xi)\,\mathrm{e}^{\pm\gamma(\xi;\eta,\mu)}\,, \tag{5.17}$$

with

$$\gamma(\xi;\eta,\mu) = 2\left[\mathrm{e}^{-\eta\xi}\left(\gamma_E + \ln(\eta|\xi|) + \sum_{n=1}^{\infty}\frac{(\eta\xi)^n}{n\cdot n!}\right) - \ln(\mu|\xi|)\right]\,. \tag{5.18}$$

The commutators (5.7-5.13) deserve some comments. In analogy with (5.2), the commutator of the left and right Cartan generators is proportional to the identity operator $\mathbb{I}$. A first novelty is the central extension multiplication factor $i\partial_{\xi_1}f(\xi_{12})$, which is different and depends on the defect. The commutation of Cartan generators with step operators reproduces the latter up to a factor which is the integral of the central extension in (5.7). Finally, the commutation of step operators leads, up to the structure functions $g_\pm$, to the bilocal operators (5.6).

It is perhaps useful to recall that the representations $\{\varrho_{\epsilon Z}\}$ of $\widehat{sl}(2)$ have a direct physical application. They describe the symmetry content of the $SU(2)$-invariant massless Thirring model with a δ-impurity. Without impurity the model has been solved long ago with bosonization by M. Halpern [32]. In the presence of a δ-defect the solution is a direct generalization of our results in the previous section.

Let us consider now the energy-momentum tensor Θ of the quantum field φ interacting with the defect [23]. The chiral components

$$\Theta_Z(x,\xi) = \theta(-x)\Theta_{-Z}(\xi) + \theta(x)\Theta_{+Z}(\xi) \tag{5.19}$$

of Θ can be expressed in terms of the generators $H_{\epsilon Z}$ by means of

$$\Theta_{\epsilon Z}(\xi) = \frac{1}{2\pi}\,:H_{\epsilon Z}\,H_{\epsilon Z}:(\xi)\,, \tag{5.20}$$

which is precisely the Sugawara representation [6]. As expected, *for fixed* $\{\epsilon,\, Z\}$ one finds

$$[\Theta_{\epsilon Z}(\xi_1)\,,\,\Theta_{\epsilon Z}(\xi_2)] = 2i\delta'(\xi_{12})\Theta_{\epsilon Z}(\xi_1) - \frac{i}{6\pi}\delta'''(\xi_{12})\,\mathbb{I}\,. \tag{5.21}$$

From the properties of $H_{\epsilon Z}$ one infers that Θ_{+R} commutes with Θ_{-L} as well as Θ_{-R} commutes with Θ_{+L}. The remaining commutators are however non-trivial. In the quasi-conformal case one finds for instance

$$[\Theta_{+R}(\xi_1)\,,\,\Theta_{+L}(\xi_2)] = i\delta'(\xi_{12})r(\lambda)\,[\Theta_{+R,\,+L}(\xi_1) + \Theta_{+L,\,+R}(\xi_1)] - \frac{ir(\lambda)^2}{6\pi}\delta'''(\xi_{12})\mathbb{I}\,, \tag{5.22}$$

where

$$\Theta_{\epsilon_1 Z_1,\,\epsilon_2 Z_2}(\xi) = \frac{1}{2\pi}\,:H_{\epsilon_1 Z_1}\,H_{\epsilon_2 Z_2}:(\xi)\,. \tag{5.23}$$

The appearance of *mixed* Sugawara terms of the type (5.23) is a new feature, which has once more its origin in the left-right and plus-minus mixing due to the defect. We observe also that the commutator (5.22) has a central term, the central charge being renormalized by a factor of $r(\lambda)^2$ with respect to (5.21). One might be tempted to change the normalization of $\Theta_{\epsilon Z}$ in order to eliminate all factors $r(\lambda)$ from the right hand side of (5.22), but then the inverse of this factor will appear in (5.21).

It is worth mentioning that the operators (5.20,5.23) close actually an algebra. A straightforward but long computation using the RT algebra relations (2.11-2.13) gives in fact

$$[\Theta_{+Z}(\xi_1)\,,\,\Theta_{+R,\,+L}(\xi_2)] = i\delta'(\xi_{12})\,[r(\lambda)\Theta_{+Z}(\xi_1) + \Theta_{+R,\,+L}(\xi_1)] - \frac{ir(\lambda)}{6\pi}\delta'''(\xi_{12})\mathbb{I} \tag{5.24}$$

and

$$\begin{aligned}&[\Theta_{+R,\,+L}(\xi_1)\,,\,\Theta_{+R,\,+L}(\xi_2)] = \\ &i\delta'(\xi_{12})\,[\Theta_{+R}(\xi_1) + r(\lambda)\Theta_{+R,\,+L}(\xi_1) + \Theta_{+L}(\xi_1)] - \frac{i[r(\lambda)^2+1]}{12\pi}\delta'''(\xi_{12})\mathbb{I}\,,\end{aligned} \tag{5.25}$$

which complete the picture in the quasi-conformal case. Like in the Kac-Moody algebra, for more general defects the commutators (5.22,5.24,5.25) involve bilocal operators of the form

$$\Theta_{\epsilon_1 Z_1,\,\epsilon_2 Z_2}(\xi_1,\xi_2) = \frac{1}{2\pi}\,:H_{\epsilon_1 Z_1}(\xi_1)\,H_{\epsilon_2 Z_2}(\xi_2):\;. \tag{5.26}$$

Summarizing, we have shown in this section how some familiar structures from conformal field theory are modified by the presence of a point-like impurity, which

preserves unitarity and locality. Together with the left-right mixing, a relevant characteristic feature is the appearance of bilocal operators.

We conclude by observing that the above construction of the $\widehat{sl}(2)$ Kac-Moody algebra and the Sugawara representation of the energy-momentum tensor can be extended to the case of $\widehat{sl}(n)$ with $n > 2$. The Klein factors, ensuring the right statistics of the Kac-Moody generators [6], are constructed in terms of the charges defined by (2.45).

6 Conclusions and perspectives

Chiral fields, vertex operators and conformal field theory on the plane played in the past two decades a fundamental role in the development of both theoretical physics and mathematics. Considering the simplest case of a massless scalar quantum field φ, we propose in the present paper a generalization which consists in a theory on the plane without a line, where dissipationless boundary conditions are imposed on φ. In physical terms the line represents the world-line of a point-like defect. The interaction of the defect with φ breaks down conformal invariance, but the breaking is fully under control and the theory is unitary. Dimensional parameters appear in the model and the left and right chiral sectors couple trough the defect. Although deformed, most of the basic structures (left and right chiral fields, vertex and Kac-Moody algebras, energy-momentum tensor,...) keep a well-defined physical and mathematical meaning. The vertex operators still carry anyon statistics and some of their correlators have anomalous dimension. Moreover, it turns out that bosonization can be successfully developed in this context. As an application of this method we solved explicitly the massless Thirring model with a generic point-like defect. We also constructed in this framework the vertex representation of the $\widehat{sl}(2)$ Kac-Moody algebra, establishing the interplay between the left and right sectors mediated by the defect. Using the Sugawara representation, we derived the main properties of energy-momentum tensor as well. The general results of our investigation have been illustrated on the concrete examples of quasi-conformal and δ-defects.

The method of bosonization with a dissipationless point-like defect, developed in this paper, suggest some new topics for research. An interesting issue, deserving further investigation, is the algebra generated by the energy-momentum tensor and the vertex operators. Concerning the physical applications, one can use our results for solving models with impurities in condensed matter physics. The simplest one is the generalization of the Luttinger model to the case with impurities. It will be also interesting to extend our framework to integrable systems with non-trivial bulk

scattering. In this case the two-body bulk scattering matrix shows up [13] as a non-trivial exchange factor in the reflection-transmission algebra. Such a generalization is expected to produce a sort of quantum deformation [33, 34] of the vertex algebra with defect.

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